Orthogonal Vector Basis Functions for Time Domain Finite Element Solution of the Vector Wave Equation

D.A. White

Lawrence Livermore National Laboratory

Center for Applied Scientific Computing

UCRL-JC-129188
December 1997

PREPRINT

This is a preprint of a paper that was submitted to the Eighth Biennial IEEE Conference on Electromagnetic Field Computation to be held in Tucson, AZ June 1-3, 1998 This preprint is made available with the understanding that it will not be cited or reproduced without the permission of the authors.

Orthogonal Vector Basis Functions for Time Domain Finite Element Solution of the Vector Wave Equation

Daniel A. White
Lawrence Livermore National Laboratory

Abstract—**In this paper we consider the solution of the vector wave equation by a discrete time vector finite element method. The popular linear edge basis functions are augmented such that the resulting capacitance matrix is diagonal, resulting in an explicit method. The accuracy and efficiency of the method is investigated via computer experiments.**

I. Introduction

The vector wave equation for the electric field is a time dependent second order partial differential equation (PDE) Application of the Galerkin procedure to the variational form of the vector wave equation results in an system of ordinary differential equations (ODE). Integrating this system of ODE's requires the solution of a large linear system (the "capacitance matrix") at every time step. When linear edge basis functions (also known as Nedelec, Whitney, and H(curl) functions) [1]–[5] are used the capacitance matrix is sparse, but non-diagonal, which is a significant drawback compared to completely explicit finite difference and finite volume methods.

The conjugate gradient (CG) algorithm has been used to solve the capacitance matrix system at every time step, with both explicit [6], [7] and implicit [8], [9] time integration methods. In [10] a variety of methods for solving the linear system were investigated, including preconditioned CG and fixed-point iterative methods. These references show that using an iterative method for the capacitance matrix system yields an accurate field solution, but that this approach is computationally intensive. Several researchers have attempted to eliminate the capacitance matrix by mass lumping or by point matching [11]–[13]. These approaches are intriguing, but the accuracy and stability properties are not well understood since they are not true Galerkin methods

In this paper a method for creating a diagonal mass matrix is presented. The linear edge basis functions are augmented such that the new basis functions are orthogonal with respect to a particular inner product. The resulting mass matrix is therefore diagonal and can be eliminated, resulting in a completely explicit method. The new set of basis functions is larger than the original set by a factor of three. The accuracy and efficiency of this approach is investigated via computer experiments on two-dimensional triangular grids.

Manuscript received December 1, 1997.
D A White, e-mail dwhite@llnl gov, fax 510-423-2993
This work was supported in part by the U S Department of Energy under Grant No. W-7405-Eng-48

II. Galerkin Formulation

We are concerned with the computer simulation of time dependent electromagnetic fields in a generic inhomogeneous volume Ω consisting of solid dielectric, magnetic, and conductive materials. There is no free charge in the volume. An appropriate PDE is the vector wave equation for the electric field $\vec{E}$

$$\epsilon\frac{\partial^2}{\partial t^2}\vec{E} = -\nabla\times\mu^{-1}\nabla\times\vec{E} - \frac{\partial}{\partial t}\vec{J} \text{ in } \Omega. \quad (1)$$

For simplicity it is assumed that the dielectric permittivity ϵ and the magnetic permeability μ are functions of position only, i e only linear, non-dispersive media will be considered The electric field on the boundary Γ is specified by

$$\hat{n}\times\vec{E} = \vec{E}_{bc} \text{ on } \Gamma, \quad (2)$$

and the two initial conditions

$$\vec{E}(t=0) = \vec{E}_{ic} \text{ in } \Omega, \quad (3)$$

$$\frac{\partial}{\partial t}\vec{E}(t=0) = \frac{\partial}{\partial t}\vec{E}_{ic} \text{ in } \Omega, \quad (4)$$

complete the description of the PDE Typically the initial conditions are zero and the problem is driven by either the time dependent current source $\vec{J}$ or the time dependent boundary condition $\vec{E}_{bc}$

The variational form of (1) is find $\vec{E}\in H(curl)$ that satisfies

$$\frac{\partial^2}{\partial t^2}\left(\epsilon\vec{E},\vec{E}^\star\right) = \left(\mu^{-1}\nabla\times\vec{E},\nabla\times\vec{E}^\star\right) - \frac{\partial}{\partial t}\left(\vec{J},\vec{E}^\star\right) \quad (5)$$

for all $\vec{E}^\star\in H_0(curl)$, where

$$(\vec{u},\vec{v}) = \int_\Omega \vec{v}\cdot\vec{u}\,d\Omega, \quad (6)$$

and

$$H_0(curl) = \{\vec{v}:\vec{v}\in H(curl), \hat{n}\times\vec{v}=0\} \quad (7)$$

In the finite element solution of (5) the space $H(curl)$ is approximated by a finite dimension subspace $W^h \subset H(curl)$ defined on a mesh, yielding a system of ODE's

$$A\frac{\partial^2}{\partial t^2}\tilde{e} = C\tilde{e} + \tilde{s}. \tag{8}$$

The variable $\tilde{e}$ is the array of degrees of freedom (DOF) and the variable $\tilde{s}$ is the array of source terms, which includes contributions from both the independent current source $\vec{J}$ and the boundary condition $\vec{E}_{bc}$. The matrix A is a symmetric positive definite matrix, with units of capacitance, which resembles the mass matrix of continuum mechanics. The matrices A and C are given by

$$A_{ij} = \left(\epsilon\vec{W}_i, \vec{W}_j\right), \tag{9}$$

$$C_{ij} = \left(\mu^{-1}\nabla\times\vec{W}_i, \nabla\times\vec{W}_j\right), \tag{10}$$

where $\vec{W}_i$ is the basis function associated with edge i.

A Original edge basis functions

The linear edge basis functions for a triangular mesh are given by

$$\vec{W}_i = N_j\nabla N_k - N_k\nabla N_j, \tag{11}$$

where N_k is the linear nodal basis function associated with node k, and edge i connects nodes j and k These functions enforce tangential continuity of fields across cell edges but allow discontinuous normal components, which is required for inhomogeneous problems. The finite element space W^h consists of the collection of all the linear edge basis functions, and therefore has dimension of N_e, the number of edges in the mesh.

B Augmented edge basis functions

As mentioned in the introduction the edge elements defined by (11) are not orthogonal. The construction of an orthogonal set of basis functions begins with the definition of a new inner product. For every cell we define

$$\langle\vec{u},\vec{v}\rangle = \sum_{i=1}^{3}\alpha_i\vec{u}(m_i)\cdot\vec{v}(m_i), \tag{12}$$

where m_i denotes the midpoint of edge i The coefficients α_i are chosen such that (12) is a second order accurate approximation to (6).

Three auxiliary basis functions are defined for every cell. These basis functions are given by

$$\vec{B}_i = N_jN_k\hat{n}_i, \tag{13}$$

where $\hat{n}_i$ is the unit normal to edge i These auxiliary basis functions have the properties:

1. zero tangential component along the the edges of the mesh,
2. each function $\vec{B}_i$ has a non-zero normal component for edge i only
3. these functions are orthogonal to each other according the the inner product (12)

Let the collection of all the auxilary basis functions be denoted by B^h, which has dimension $3N_c$ where N_c is the number of cells in the mesh.

Using the auxilary basis functions (13), new edge basis functions $\hat{Z}$ are defined by

$$\vec{Z}_i = \vec{W}_i - \sum_{i=1}^{3}\frac{\left\langle\vec{W}_i, \vec{B}_j\right\rangle}{\left\langle\vec{B}_j, \vec{B}_j\right\rangle} \tag{14}$$

within each cell. Due to property 1, these functions have the same tangential components on cell edges as the original edge elements, hence continuity of tangential field components is preserved. Due to property 2, the new edge basis functions $\hat{Z}$ have zero normal component on the cell edges Therefore every $\vec{Z}_i$ is orthogonal to every $\vec{B}_j$. Since the original $\vec{W}_i$ functions have non-zero tangential component on edge i only, the new edge basis functions $\vec{Z}_i$ are orthogonal to each other as well. Let the collection of all the new edge basis functions $\vec{Z}$ be denoted by Z_h.

C Explicit system of ODE's

Using our inner product (12), we define an approximate variational form of the vector wave equation

$$\frac{\partial^2}{\partial t^2}\left\langle\epsilon\vec{E}, \vec{E}^\star\right\rangle = \left\langle\mu^{-1}\nabla\times\vec{E}, \nabla\times\vec{E}^\star\right\rangle - \frac{\partial}{\partial t}\left\langle\vec{J}, \vec{E}^\star\right\rangle. \tag{15}$$

The electric field is constrained to be in the space $F^h = B^h \cap Z^h$, and the test space is given by $F_0^h = \{\vec{v} : \vec{v} \in F^h, \hat{n}\times\vec{v} = 0\}$. This results in a new system of ODE's

$$D\frac{\partial^2}{\partial t^2}\tilde{e} = K\tilde{e} + \tilde{s}. \tag{16}$$

where the matrices are given by

$$D_{ij} = \langle\epsilon F_i, Fj\rangle \tag{17}$$

$$K_{ij} = \langle\mu^{-1}\nabla\times F_i, \nabla\times F_j\rangle \tag{18}$$

Since D is a diagonal matrix, (16) can be easily integrated using the leapfrog method

$$\tilde{e}^{(n+1)} = \left(2I + (\Delta t)^{-2}D^{-1}K\right)\tilde{e}^n - \tilde{e}^{n-1}. \tag{19}$$

In the following Equations (11)-(19) will be referred to as the orthogonal finite element method

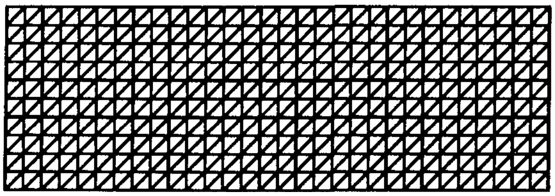

Fig 1 An example 600 cell grid of 1 by 3 rectangular cavity

III. Results

In this section the orthogonal finite element method described above is validated by comparing computed results to analytical results. The first experiment consists of a $1m$ by $1/3m$ rectangular cavity with perfectly conducting walls. The electric field is confined to the $x-y$ plane and the magnetic field is aligned in the z direction. This is often referred to as a TE_z mode. The speed of light is set to unity for convenience. A time domain method can be used to compute the resonant frequencies of a cavity by exciting the cavity with a pulse and evolving the electric field in time. The amplitude of the electric field along a selected edge is is stored for every time step. This signal can then be multiplied by a suitable window function and the Fourier transformed to yield the power spectrum of the signal. The peaks in the power spectrum are the resonant frequencies of the cavity.

The rectangular cavity is modeled using a sequence of triangular grids. An example grid is shown in Fig. 1. For each grid the electric field is evolved for $100s$, with the time step chosen such that the method is stable. The CPU time for the orthogonal finite element method is compared to that for the original method in Table I. It is interesting to note that using the orthogonal basis functions increased the number of DOF by a factor of three (approximately) but reduced the CPU time by a factor of three (approximately). The capacitance matrix for the original method was solved using diagonal preconditioned conjugate gradient, with a maximum of 15 iterations.

As a second example we consider the resonant frequencies of a circular cavity of radius $1m$. As with the rectangular cavity above, we are interested in the case of perfectly conducting walls and transverse electric fields. The circular cavity was modeled using a sequence of triangular grids. An example grid is shown in Fig. 2. The simulation procedure is the same as that for the rectangular cavity The CPU time for the orthogonal finite element method is compared to that for the original method in Table II Again, the CPU time is reduced by a factor of three even though the number of DOF is increased.

TABLE I

Rectangular cavity CPU time

	Original		Orthogonal	
Cells	DOF	CPU	DOF	CPU
600	245	84.9	695	34.4
2400	940	863.5	2740	272.2
5400	2085	2989.8	6135	972.9
9600	3680	7173.4	10880	2166.9

TABLE II

Rectangular cavity CPU time

	Original		Orthogonal	
Cells	DOF	CPU	DOF	CPU
74	101	3 45	323	1 68
296	424	34.58	1312	14 18
904	1316	241.66	4028	91 52
3592	5308	9725.1	16084	3249.7

The above results indicate that the use of orthogonal vector basis functions can significantly decrease the required CPU time for a given calculation. It is also important to discuss the accuracy of the method. The time domain vector finite element method with the original linear edge basis functions is known to have a second order accurate numerical dispersion relation for general triangular grids [14]. The accuracy of the numerical dispersion relation is equal to the rate of convergence of computed resonant frequencies of cavities, hence cavity simulations can be used to determine the accuracy of a time domain method.

For the rectangular cavity described above the frequency of the TE_{14} mode is 2.5 Hz. The logarithm of the error of this frequency is shown in Fig. 3 versus the logarithm of the cell size (lower curve). The slope of the least-square fit line is -2 04, indicating that the resonant frequency calculation is converging at second order.

For the circular cavity described above the frequency of the TE_{11} mode is 0.29303 Hz. The logarithm of the error of this frequency is shown in Fig. 3 versus the logarithm of the cell size (upper curve). The slope of the least-square fit line is -1.98, indicating that the resonant frequency calculation is again converging at second order.

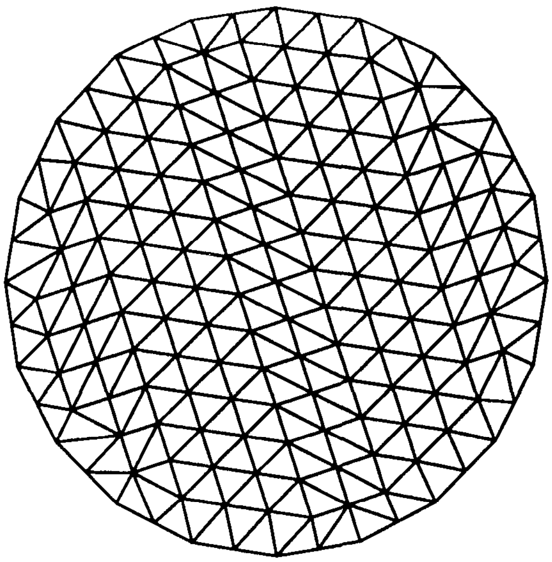

Fig 2. An example 296 cell grid of circular cavity

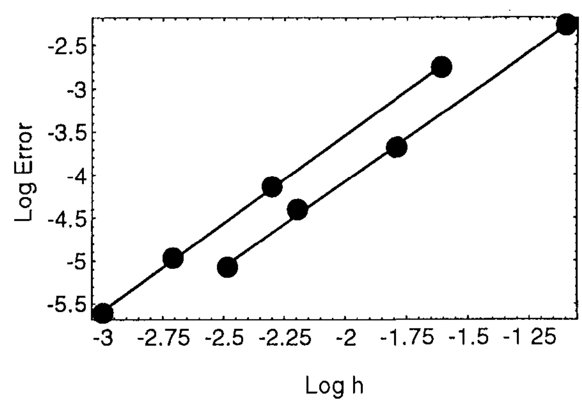

Fig. 3. Error of resonant frequency calculation indicates second order accuracy

IV. Conclusions

An approximate variational form of the vector wave equation is solved using the Galerkin formulation. The approximate variational form uses an new inner product which is a second order accurate approximation of the traditional inner product. The original linear edge basis functions are augmented such that the new basis functions are orthogonal with respect this the new inner product. This procedure guarantees a diagonal capacitance matrix While the new system of ODE's is larger than the original system, it can be integrated significantly faster the original system For a given grid the orthogonal finite element method is approximately three times faster than the original method. The orthogonal basis function method has the same second order accuracy as the original method.

V. Acknowledgements

The author would like to thank L. Cowsar for useful discussions and pointers to the literature.

References

[1] Nedelec, J C , "Mixed finite elements in R3," *Numer Math* , 35, pp 315–341, 1980.

[2] Bossavit A., "Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism," *IEE Proceedings*, v. 135, pt A, n 8, pp 493–500, 1988.

[3] Bossavit A and Mayergoyz I , "Edge elements for scattering problems," *IEEE Trans. Mag.*, v 25, n 4, pp 2816–2821, 1989.

[4] Cendes Z J , "Vector finite elements for electromagnetic field problems," *IEEE Trans Mag* , v 27, n 5, pp. 3958–3966, 1991

[5] Lee J F , Sun D. K., and Cendes Z J , "Tangential vector finite elements for electromagnetic field computation," *IEEE Trans Mag.*, v. 27, n 5, pp 4032–4035, 1991.

[6] Lee J F , "WETD - A finite element time domain approach for solving Maxwell's equations," *IEEE Microwave and Guided Wave Letters*, v 4, n 1, pp 11–13, 1994

[7] Mahadevan K , Mittra R. and Vaidya P. M., "Use of Whitney's edge and face elements for efficient finite element time domain solution of Maxwell's equations," *J. of Electromagnetic Waves and Appl.*, v 8, n 9/10, pp 1178–1191, 1994.

[8] Lee J. F and Sacks Z., "Whitney elements time domain (WETD) methods," *IEEE Trans. on Magnetics*, v 31, n 3, pp 1325–1329, 1995

[9] Gedney S D. and Navsariwala U , "An unconditionally stable finite element time domain method of the vector wave equation," *IEEE Microwave and Guided Wave Letters*, v 5, n 10, pp. 332–334, 1995

[10] White D. A , "Discrete time vector finite element methods for solving Maxwell's equations on 3D unstructured grids," PhD Dissertation, University of California at Davis, 1997.

[11] Feliziani M. and Maradei F , "Point matched finite element time domain method using vector elements," *IEEE Trans Magnetics*, v. 30, n 5, pp. 3184–3187, 1994

[12] Thng C. H., Booton R C , and Gupta K C , "FDTD compatible edge-element, face-element time domain method for electromagnetics," *Int J Microwave and Millimeter Wave CAE*, v 6, n. 1, pp. 69–77, 1996

[13] Lee J F , Lee R and Cangellaris A , "Time domain finite element methods," *IEEE Trans Antennas and Propagation*, v. 45, n 3, pp 430–442, 1997.

[14] White D and Rodrigue G , "Improved vector FEM solutions of Maxwell's equations using grid preconditioning," *Int J Num Meth. Eng.*, v 40, n. 20, pp 3815–3837, 1997